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Some improved two-stage shrinkage testimators for the mean of normal distribution

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Abstract

In this paper, we introduced some two-stage shrinkage testimators (TSST) for the mean μ when a prior estimate μ_0 of the mean μ is available from the past, by considering a feasible form of the shrinkage weight function which is used in both of the estimation stages with different quantities. The expressions for the bias, mean squared error, expected sample size and relative efficiency for the both cases when σ^2 known or unknown, are derived and studied. The discussion regarding the usefulness of these testimators under different situations is provided as conclusions from various numerical tables obtained from simulation results.

MSC: Primary 62F10, Secondary 62F99.

Keywords: Normal distribution, two-stage, shrinkage, preliminary test region, relative efficiency.

1 Introduction

1.1 TSST and Background

Let X be normally distributed with unknown mean μ and variance σ^2. Assume that prior information about μ is available in the form of an initial estimate μ_0 of μ. However, in certain situations the prior information is available only in the form of an initial guess value μ_0 of μ, then this guess may be utilized to improve the estimation procedure. For example, a bulb producer may know that the average life of his product may be close to 1000 hours. Here we may take $\mu_0 = 1000$. In such a situation it is natural to start

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with an estimator $\bar{X}$ of μ and modify it by moving it closer to μ_0, so that the resulting estimator, though perhaps biased, has a smaller mean squared error than that of $\bar{X}$ in some interval around μ_0. This method of constructing an estimator of θ that incorporates the prior information μ_0 leads to what is known as a shrunken estimator (see Thompson, 1968).

At the same time, it is an important aspect of estimation that one should be able to get an estimator quickly using minimum cost of experimentation. The cost of experimentation can be achieved by using any prior information available about μ and devising a two-stage shrunken estimator in which it is possible to obtain an estimator from a small first stage sample, and an additional second stage sample is required only if this estimator is not reliable (see Kambo, Handa and Al-Hemyari, 1991). The earliest work on two-stage estimation procedure is the paper by Katti (Katti, 1962). He developed a two-stage technique for the mean (μ) of a normal population when the variance (σ^2) is known. A number of other authors (see Al-Hemyari, 2009; Al-Hemyari and Al-Bayyati, 1981; Arnold and Al-Bayyati, 1970; Kambo *et al.*, 1992; Kambo *et al.*, 1991; Waiker, Ratnaparkhi, Schuurmann, 2001; Ratnaparkhi, Waiker, Schuurmann, 2001 and Waiker, Schuurmann and Raghunathan, 1984) have tried to develop new two-stage shrinkage testimators of the Katti type. The relevance of such types of TSST lies in the fact that, though perhaps they are biased, have smaller MSE than $\bar{X}$ in some interval around μ_0. A Two-stage shrinkage testimation (TSST) procedure is defined as follows. Let $X_{1i},\ i = 1,2,\ldots,n_1$ be a random sample of small size n_1 from $f(x|\mu)$. Compute the sample mean $\bar{X}_1$ and sample variance s^2 (unbiased estimator of σ^2, if σ^2 is unknown) based on n_1 observations. Construct a preliminary test region (R) in the space of μ, based on μ_0 and an appropriate criterion. If $\bar{X}_1 \in R$, shrink $\bar{X}_1$ towards μ_0 by shrinkage factor $0 \le \varphi(\bar{X}_1) \le 1$ and use the estimator $\varphi(\bar{X}_1)(\bar{X}_1 - \mu_0) + \mu_0$ for μ. But if $\bar{X}_1 \notin R$, obtain $X_{2i},\ i = 1,2,\ldots,n_2$ an additional sample of size $n_2 (= n - n_1)$, compute $\bar{X}_2$, and take the estimator of μ as the combined sample mean $\bar{X} = (n_1\bar{X}_1 + n_2\bar{X}_2)/(n_1 + n_2)$. Thus a two-stage shrinkage testimator of μ is given by:

$$\hat{\mu} = \{[\varphi(\bar{X}_1)(\bar{X}_1 - \mu_0) + \mu_0]I_R + [\bar{X}]I_{\bar{R}}\}, \tag{1}$$

where I_R and $I_{\bar{R}}$ are respectively the indicator functions of the acceptance region R and the rejection region $\bar{R}$.

1.2 The Modification

The TSST $\hat{\mu}$ is completely specified if the shrinkage weight factor $\varphi(\bar{X}_1)$ and the region R are specified. Consequently, the success of $\hat{\mu}$ depends upon the proper choice of $\varphi(\bar{X}_1)$ and R. Some choices for $\varphi(\bar{X}_1)$ and R are given in Al-Hemyari, 2009; Al-Hemyari and Al-Bayyati, 1981; Arnold and Al-Bayyati, 1970; Kambo *et al.*, 1992; Kambo *et al.*, 1991; Katti, 1962; Waiker *et al.*, 2001; Ratnaparkhi *et al.*, 2001 and Waiker *et al.*, 1984.

Other choices with different estimation problems are discussed in Al-Hemyari, Kurshid and Al-Gebori, 2009; Al-Hemyari and Al-Bayyati, 1981; Saxena and Singh, 2006 and Thompson, 1968. We proposed two-stage shrinkage testimators in this paper for the mean μ when σ^2 is known or unknown denoted by $\tilde{\mu}_i,\ i=1,2$, which are a modifications of $\hat{\mu}$ defined in (1). The proposed testimator takes the general form:

$$\tilde{\mu}_{=}\{[\varphi(\bar{X}_1)(\bar{X}_1-\mu_0)+\mu_0]I_R+[[(1-\varphi(\bar{X}_1)(\bar{X}-\mu_0)+\mu_0]]I_{\bar{R}}\}. \tag{2}$$

The main distinguishing feature of this type of TSST from conventional two stage shrinkage testimators is that, the pretest region rejects the prior estimate μ_0 only partially and even if $\bar{X}_1 \notin R,\ \mu_0$, is given some weight though small in estimation of second stage. The expressions for the bias, mean squared error, expected sample size and relative efficiency of $\tilde{\mu}$ for the both cases when σ^2 known or unknown, are derived and studied theoretically and numerically. Comparisons with the earlier known results are made.

2 Formulation, assumptions and derivation of the proposed TSST with known σ^2

We define the general proposed estimator when σ^2 is known in this section. The bias, mean squared error, expected sample size, and relative efficiency expressions of the proposed testimator are derived. A suitable shrinkage function $\varphi(\bar{X}_1)$ is chosen, and finally some properties are also discussed.

2.1 The proposed testimator

Let X be normally distributed with unknown μ and known variance σ^2. Assume that a prior estimate μ_0 about μ is available from the past. The first proposed testimator is:

$$\tilde{\mu}_1=\{[\bar{X}_1-ae^{-n_1b(\bar{X}_1-\mu_0)^2/\sigma^2}(\bar{X}_1-\mu_0)]I_R+[[ae^{-n_1b(\bar{X}_1-\mu_0)^2/\sigma^2}(\bar{X}-\mu_0)+\mu_0]]I_{\bar{R}}\}. \tag{3}$$

R_1 is taking as the pretest region of size α for testing $H_0:\mu=\mu_0$ against $H_1:\mu\neq\mu_0$, where

$$R_1=[\mu_0-z_{\alpha/2}\sigma/\sqrt{n_1},\mu_0+z_{\alpha/2}\sigma/\sqrt{n_1}],\ \varphi(\bar{X}_1)=1-a\exp[-n_1b(\bar{X}_1-\mu_0)^2/\sigma^2], \tag{4}$$

$b\geq 0,\ 0\leq a\leq 1$, and $z_{\alpha/2}$ is the upper $100(\alpha/2)$ percentile point of the standard normal distribution.

2.2 Bias ratio, MSE, Expected sample size and Relative Efficiency Expressions

It can be easily shown that the bias and mean squared error of $\tilde{\mu}_1$ are, respectively, given by:

$$\begin{aligned} B(\tilde{\mu}_1|\mu) &= (\sigma/\sqrt{n_1})\{J_1(a_1,b_1)+\lambda_1(J_0(a_1,b_1)-1)+a(2b+1)^{-3/2}e^{-b\lambda_1^2/(2b+1)}\times \\ &\times\ ((1/(1+f))-a(1(f+1)^{-1})[\sqrt{2b+1}J_1(a_2,b_2)+\lambda_1 J_0(a_2,b_2)]) \\ &-\ a\lambda_1\sqrt{f}(1+f)^{-1}\sqrt{2b+1}e^{-b\lambda_1^2/(2b+1)}(J_0(a_2,b_2)-1)\}, \end{aligned} \tag{5}$$

$$\begin{aligned} MSE(\tilde{\mu}_1|\mu) &= (\sigma^2/n)\{J_2(a_1,b_1)-2a(2b+1)^{-5/2}e^{-b\lambda_1^2/(2b+10)}[(2b+1)J_2(a_2,b_2) \\ &+\ \lambda_1(1-2b)\sqrt{2b+1}J_1(a_2,b_2)-2b\lambda_1^2 J_0(a_2,b_2)]+a^2(1-(1+f)^{-2})\times \\ &\times\ (4b+1)^{-5/2}e^{-2b\lambda_1^2/(4b+1)}[(4b+1)J_2(a_3,b_3)+2\lambda_1\sqrt{4b+1}J_1(a_3,b_3) \\ &+\ \lambda_1^2 J_0(a_3,b_3)]+\lambda_1^2(1-J_0(a,b))+a^2(1+f)^{-2}(4b+1)^{-5/2}((4b+1) \\ &+\ \lambda_1^2)e^{-2b\lambda_1^2/(4b+1)}-2a\lambda_1(1+f)^{-1}(2b+1)^{-3/2}e^{-b\lambda_1^2/(2b+1)}\times \\ &\times\ [\lambda_1-\sqrt{2b+1}J_1(a_2,b_2)-\lambda_1 J_0(a_2,b_2)]+a^2 f^2(1+\lambda_1^2)(1+f)^{-2}\times \\ &\times\ (4b+1)^{-1/2}e^{-2b\lambda_1^2/(4b+1)}(1-J_0(a,b))+2a^2\sqrt{f}(1+f)^{-2}\times \\ &\times\ (4b+1)^{-3/2}e^{-2b\lambda_1^2/(4b+1)}[\lambda_1+\sqrt{4b+1}J_1(a_2,b_2)+\lambda_1 J_0(a_2,b_2)] \\ &\quad -2a\lambda_1^2\sqrt{f}(1+f)^{-1}(2b+1)^{-1/2}e^{-b\lambda_1^2/(2b+1)}J_0(a_2,b_2)\}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} a_1 &= \lambda_1 - z_{\alpha/2}, & b_1 &= \lambda_1 + z_{\alpha/2}, \\ a_2 &= (\lambda_1^{-} z_{\alpha/2})/\sqrt{2b+1}, & b_2 &= (\lambda_2 + z_{\alpha/2})/\sqrt{2b+1}, \\ a_3 &= (\lambda_1 - z_{\alpha/2})/\sqrt{4b-1}, & b_3 &= (\lambda_1 + z_{\alpha/2})/\sqrt{4b-1}, \\ \lambda_1 &= \sqrt{n_1}(\mu-\mu_0)/\sigma, & f &= n_2/n_1, \end{aligned}$$

and

$$J_i(a_j,b_j) = \int_{a_j}^{b_j} \frac{1}{\sqrt{2\pi}} y^i e^{-y^2/2} dy, \quad i=0,1,2,\ j=1,2. \tag{7}$$

The expected sample size and the efficiency of $\tilde{\mu}_1$ relative to $\bar{X}$ are given respectively by:

$$E(n|\tilde{\mu}_1) = n_1[1 + f(1 - J_0(a_1, b_1)), \quad (8)$$

$$Eff(\tilde{\mu}_1|\mu) = \sigma^2/E(n|\tilde{\mu}_1)MSE(\tilde{\mu}_1|\mu). \quad (9)$$

2.3 Selection of '*a*'

It seems reasonable to select 'a' that minimizes the $MSE(\tilde{\mu}_1|\mu_0)$. Setting $((\partial/\partial a)$ $MSE(\tilde{\mu}_1|\mu_0))$ to zero, we get:

$$\begin{aligned} a = \bar{a}_1 = (1/n_1)[(2b+1)^{-3/2}J_2(a_2^*, b_2^*)/((4b+1)^{-3/2}((1+f)^{-2}(1 - J_2(a_3^*, b_3^*))+ \\ + J_2(a_3^*, b_3^*)) + \alpha f(1+f)^{-2}/\sqrt{4b+1})], \end{aligned} \quad (10)$$

where

$$\begin{aligned} a_2^* &= -z_{\alpha/2}/\sqrt{2b+1}, & b_2^* &= -a_2^*, \\ a_3^* &= -z_{\alpha/2}/\sqrt{4b-1}, \quad \text{and} & b_3^* &= -a_3^*. \end{aligned}$$

Since $(\partial^2/\partial a^2)MSE(\tilde{\mu}_1|\mu_0) \geq 0$. It follows that the minimizing value of $a \in [0,1]$ is given by:

$$\tilde{a} = \begin{cases} 0, \text{ if } & \bar{a}_1 \leq 0, \\ \bar{a}_1, \text{ if } & 0 \leq \bar{a}_1 \leq 1, \\ 1, \text{ if } & \bar{a}_1 \geq 1. \end{cases} \quad (11)$$

2.4 Some properties

i) Unbiasedness: If $\mu = \mu_0$, or $n_1 \to \infty$, the proposed testimator turns into the unbiased estimator, otherwise it is biased. Thus, we conclude the following: There does not exist, any unbiased estimator of μ in the class of testimators $\{\tilde{\mu} : 0 \leq \varphi(\bar{X}_1) \leq 1\}$ except the above undesirable cases.

ii) Minimum mean squared error estimator: It is not easy with the type of the proposed testimator to establish the minimum mean squared error biased estimator, i.e., $MSE(\tilde{\mu}|\mu) \leq MSE(\bar{X})$, for every $\varphi(\bar{X}_1)$ and every μ with strict inequality for at least one μ. But when $\mu = \mu_0$ the inequality holds, this means that by a proper choice of $\varphi(\bar{X}_1)$, the proposed TSST performs better (in the sense of smaller MSE) than $\bar{X}$ in the neighbourhood of μ_0. Also $Eff(\tilde{\mu}_1|\mu) \geq 1$ as $\lambda_1 \to \pm\infty$.

iii) Odd and even functions: It is easily seen that $B(\tilde{\mu}_1|\mu)$ is an odd function of λ_1, whereas $E(n|\tilde{\mu}_1)$, $MSE(\tilde{\mu}_1|\mu)$ and $Eff(\tilde{\mu}_1|\mu)$ are all even functions of λ_1.

iv) Consistent and dominant estimator: since

$$\lim_{n_1\to\infty} B(\tilde{\mu}_1|\mu) = 0 \quad \text{and} \quad \lim_{n_1\to\infty\infty} MSE(\tilde{\mu}_1|\mu) = 0,$$

$\tilde{\mu}_1$ is a consistent estimator of μ. Also $\tilde{\mu}_1$ dominates $\bar{X}$ in large n_1 and n_2 in the sense that

$$\lim_{n_1,n_2\to\infty} [MSE(\tilde{\mu}_1|\mu) - MSE(\bar{X})] \le 0.$$

v) Special cases: It may be noted here, when $a = 0$, the equations (3), (5), (6), (8) & (9) agree with the result of Katti (Katti, 1962) also when $b = 0$, $(1-a) = k$, the same expressions agree with the result of Arnold and Al-Bayyati (Arnold and Al-Bayyati, 1970) when $b \to \infty$ and $a = 1$, the result agrees with the result of Kambo, Handa and Al-Hemyari (Kambo *et al.*, 1991), and when the second stage shrinkage function $(1-\varphi(\bar{X}_1)) = 1$, the result agrees with the result of Al-Hemyari (Al-Hemyari, 2009).

3 Formulation, assumptions and derivation of the proposed TSST with unknown σ^2

3.1 The proposed testimator

When σ^2 is unknown, it is estimated by

$$s^2 = \sum_{i=1}^{n_1} (X_i - \bar{X}_1)^2/(n_1-1).$$

Again taking region R_2 as the pretest region of size α for testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ in the testimator $\tilde{\mu}_1$ defined in equation (3) and denoting the resulting estimator as $\tilde{\mu}_2$. The testimator $\tilde{\mu}_2$ employs R_2 given by:

$$\begin{aligned} R_2 &= [\mu_0 - t_{\alpha/2,n_1-1}s/\sqrt{n_1},\, \mu_0 + t_{\alpha/2,n_1-1}s/\sqrt{n_1}], \\ \varphi(\bar{X}_1) &= 1 - a\exp[-n_1 b(\bar{X}_1-\mu_0)^2/s^2], \end{aligned} \tag{12}$$

where $t_{\alpha/2,n_1-1}$ is the upper $100(\alpha/2)$ percentile point of the t distribution with n_1-1 degrees of freedom.

3.2 Bias ratio, MSE, expected sample size and relative efficiency expressions

The expressions for bias, MSE and expected sample size are given respectively by:

$$\begin{aligned} B(\tilde{\mu}_2|\mu) = {} & (\sigma/\sqrt{n_1})\int_{s^2=0}^{\infty}\{J_1(a_1,b_1)+\lambda_1(J_0(a_1,b_1)-1)+a(2b+1)^{-3/2}\times \\ & \times\ e^{-b\lambda_1^2/(2b+1)}((1/(1+f))-a(1+(f+1)^{-1})[\sqrt{2b+1}J_1(a_2,b_2) \\ & +\ \lambda_1 J_0(a_2,b_2)])-a\lambda_1\sqrt{f}(1+f)^{-1}\sqrt{2b+1}e^{-b\lambda_1^2/(2b+1)}\times \\ & \times\ (J_0(a_2,b_2)-1)\}f(s_1^2|\sigma^2)ds_1^2, \end{aligned} \tag{13}$$

$$\begin{aligned} MSE(\tilde{\mu}_2|\mu) = {} & (\sigma^2/n)\int_{s_1^2=0}^{\infty}\{J_2(a_1,b_1)-2a(2b+1)^{-5/2}e^{-b\lambda_1^2/(2b+10)}\times \\ & \times\ [(2b+1)J_2(a_2,b_2)+\lambda_1(1-2b)\sqrt{2b+1}J_1(a_2,b_2) \\ & -\ 2b\lambda_1^2 J_0(a_2,b_2)]+a^2(1-(1+f)^{-2})(4b+1)^{-5/2}e^{-2b\lambda_1^2/(4b+1)}\times \\ & \times\ [(4b+1)J_2(a_3,b_3)+2\lambda_1\sqrt{4b+1}J_1(a_3,b_3)+\lambda_1^2 J_0(a_3,b_3)] \\ & +\ \lambda_1^2(1-J_0(a,b))+a^2(1+f)^{-2}(4b+1)^{-5/2}((4b+1)+\lambda_1^2)\times \\ & \times\ e^{-2b\lambda_1^2/(4b+1)}-2a\lambda_1(1+f)^{-1}(2b+1)^{-3/2}e^{-b\lambda_1^2/(2b+1)}\times \\ & \times\ [\lambda_1-\sqrt{2b+1}J_1(a_2,b_2)-\lambda_1 J_0(a_2,b_2)]+a^2f^2(1+\lambda_1^2)(1+f)^{-2}\times \\ & \times\ (4b+1)^{-1/2}e^{-2b\lambda_1^2/(4b+1)}(1-J_0(a,b))+2a^2\sqrt{f}(1+f)^{-2}\times \\ & \times\ (4b+1)^{-3/2}e^{-2b\lambda_1^2/(4b+1)}[\lambda_1+\sqrt{4b+1}J_1(a_2,b_2)+\lambda_1 J_0(a_2,b_2)] \\ & -2a\lambda_1^2\sqrt{f}(1+f)^{-1}(2b+1)^{-1/2}e^{-b\lambda_1^2/(2b+1)}J_0(a_2,b_2)\}f(s_1^2|\sigma^2), \end{aligned} \tag{14}$$

and

$$E(n|\tilde{\mu}_2) = n_1\int_0^{\infty}[1+f(1-J_0(a_1^*,b_1^*))]f(s^2|\sigma^2)ds^2, \tag{15}$$

where $a_3^* = (\lambda_1 - t_{\alpha/2,n_1-1}s_1/\sigma)/\sqrt{4b+1}$, $\quad b_3^* = (\lambda_1 + t_{\alpha/2,n_1}s_1/\sigma)/\sqrt{4b+1}$,
$a_2^* = (\lambda_1 - t_{\alpha/2,n_1-1}xxs_1/\sigma)/\sqrt{2b+1}$, $\quad b_2^* = (\lambda_1 + t_{\alpha/2,n_1-1}s_1/\sigma)/\sqrt{2b+1}$,
$a_1^* = \lambda_1 - t_{\alpha/2,n_1-1}s_1/\sigma$, $\quad b_1^* = \lambda_1 + t_{\alpha/2,n_1-1}$,

and $f(s^2|\sigma^2)$ is the p. d .f. of s^2. If $\mu = \mu_0$, the above expressions reduce to:

$$B(\tilde{\mu}_2|\mu_0) = 0, \tag{16}$$

$$
\begin{aligned}
MSE(\tilde{\mu}_2|\mu_0) = & (\sigma^2/n_1)\{(1-\alpha)((1-2a(2b+1)^{-3/2}+a^2(4b+1)^{-3/2}\times \\
& \times (1-(1+f)^{-2}))+a^2(1+f)^{-2}(4b+1)^{-3/2}-2t_{\alpha/2,n_1-1}\times \\
& \times \Gamma(n_1/2)/g_0\sqrt{\pi(n_1-1)}+4at_{\alpha/2,n_1-1}\Gamma(n_1/2)/[g_2\sqrt{\pi(n_1-1)}\times \\
& \times (2b+1)+2a^2(1-(1+f)^{-2})t_{\alpha/2,n_1-1}\Gamma(n_1/2) \\
& \quad /[g_4\sqrt{\pi(n_1-1)}(4b+1)]+\alpha a^2 f\sqrt{4b+1}(1+f)^{-2}\},
\end{aligned}
\tag{17}
$$

where $g_m = [\Gamma((n_1-1)/2)(1+t^2_{\alpha/2,n_{1-1}}(mb+1)/(n_1-1))^{n_1/2}]$ $m=0,2,4$ and

$$E(n|\tilde{\mu}_2) = n_1[1+\alpha f]. \tag{18}$$

The relative efficiency of $\tilde{\mu}_2$ defined by:

$$Eff(\tilde{\mu}_2|\mu_0) = \sigma^2/E(n|\tilde{\mu}_2)MSE(\tilde{\mu}_2|\mu_0) \tag{19}$$

Also, it is easily seen that

$$\lim_{n_1,n_2\to\infty} MSE(\tilde{\mu}_2|\mu) = 0 \quad \text{and} \quad \lim_{n_1,n_2\to\infty}[MSE(\tilde{\mu}_2|\mu_0)-MSE(\bar{X})] \le 0.$$

3.3 Selection of ‘a’

Proceeding in the manner as in the last section, we get the minimizing value of ‘a’ as follows:

$$
\begin{aligned}
a = \bar{a}_2 = & (1/n_1)[(1-\alpha)(2b+1)^{-3/2}-2(t_{\alpha/2,n_1-1}\Gamma(n_1/2)/\sqrt{\pi(n_1-1)}(2b+1)g_2) \\
& /[(4b+1)^{-3/2}((1-(1+f)^{-2})(1-\alpha)+(1+f)^{-2})+2(t_{\alpha/2,n_1-1}\Gamma(n_1/2) \\
& /\sqrt{\pi(n_1-1)}(4b+1)^{-1}g_4(1-(1+f)^{-2})+\alpha f(1+f)^{-2}(4b+1)^{-1/2}).
\end{aligned}
\tag{20}
$$

Since $(\partial^2/\partial a^2)MSE(\tilde{\mu}_2|\mu_0) \ge 0$. It follows that the minimizing value of $a \in [0,1]$ is given by:

$$
\tilde{a} = \begin{cases} 0, \text{ if } & \bar{a}_2 \le 0 \\ \bar{a}_2, \text{ if } & 0 \le \bar{a}_2 \le 1, \\ 1, \text{ if } & \bar{a}_2 \ge 1. \end{cases} \tag{21}
$$

4 Examples

Example 1: Data were collected regarding weight, length and diameter of the Carp fish in Dokan lake (see Al-Hemyari and Al-Bayyati, 1981), where the estimation of the hunted quantity was calculated. In this example we will use the same data to illustrate how we can apply the proposed testimator $\tilde{\mu}_1$ as an estimator for the average length of the Carp fish. From the past data we had $\mu_0 = 33.314$, and $\sigma^2 = 13.814$. We draw a sample of size $n_1 = 5, 10$, $\bar{X}_1, R_1$ and $\tilde{\mu}_1$ are computed and given below for a number of values assigned for $n_2 = 10, 20, 30, 40$, $\alpha = 0.01$, and $b = 0.001$. The corresponding values of $Eff(\tilde{\mu}_1|\mu)$, $(\sqrt{n_1}/\mu)B(\tilde{\mu}_i|\mu)$, $E(n|\tilde{\mu}_i)$, $pr\{\bar{X}_1 \in R_1)$, $E(n|\tilde{\mu}_1)/n$, and $100(n_2/n)pr\{\bar{X}_1 \in R_1\}$ can be obtained from the Tables 1-6 using the corresponding constants $f = n_2/n_1$ and λ.

n_1	$\bar{X}_1$	$R_1 = [a,b]$	$n_2 = 5$	$n_2 = 10$	$n_2 = 20$	$n_2 = 30$	$n_2 = 40$
5	36.700	29.197,37.595	34.67	34.33	33.99	33.66	33.34
10	34.400	28.038,34.092	33.75	33.64	33.53	33.42	33.32

Example 2: Another data set will be used here to illustrate the calculations of the second proposed testimator $\tilde{\mu}_2$. An instructor is teaching a statistics course for many years at Nizwa University. Three groups of 120 students were registered in this course (cohort 2008) and all the students appeared for the final test. The teacher wants to estimate the average of the final score test using the prior value $\mu_0 = 82.19$ (from the last year test), and he decided the following: if $\tilde{\mu}_1 > \bar{X}_1$, he will consider $\tilde{\mu}_1$ as the sample mean of the current data and then he will modify the student's result on this basis. Based on a sample of size $n_1 = 5, 11$, $\bar{X}_1$, s, R_2 and $\tilde{\mu}_2$ are computed for a number of values assigned for $n_2 = 5, 11, 20, 35, 44$, $\alpha = 0.01$, $b = 0.001$ and given below. Some values of $Eff(\tilde{\mu}_2|\mu_0)$, $E(n|\tilde{\mu}_2)$ and $(100(n_2/n)pr\{\bar{X}_1 \in R_2\})$ are presented in Tables 7 and 8.

n_1	$\bar{X}_1$	s	$R_2 = [a,b]$	$n_2 = 5$	$n_2 = 11$	$n_2 = 20$	$n_2 = 35$	$n_2 = 44$
5	74.182	6.780	68.229,96.151	78.95	79.75	80.54	81.34	82.13
11	80.800	9.478	73.134,91.246	81.63	81.77	81.91	82.05	82.19

5 Simulation, Empirical results and Conclusions

A natural way of comparing the proposed two-stage shrinkage testimator is to study its performance with respect to the classical *MLE* $\bar{X}$ and with existing testimators given in Al-Hemyari, 2009; Arnold and Al-Bayyati, 1970; Kambo *et al.*, 1991; Katti, 1962; Waiker, Ratnaparkhi and Schuurmann, 2001; Ratnaparkhi *et al.*, 2001 and Waiker *et al.*, 1984. The comparisons were done on the basis of many properties and different criterion. The computations of $Eff(\tilde{\mu}_i|\mu)$, $(\sqrt{n_1}/\mu)B(\tilde{\mu}_i|\mu)$, $E(n|\tilde{\mu}_i)$, probability of avoiding the second stage sample $(pr\{\bar{X}_1 \in R_i\})$, the ratio $E(n|\tilde{\mu}_i)/n$, the percentage of overall sample saved $(100(n_2/n)pr\{\bar{X}_1 \in R_i\})$, were done for the two-stage shrinkage testimators $\tilde{\mu}_1$ and $\tilde{\mu}_2$. From expressions (4, 5, 6, 8, 9, 11), it is observed that $Eff(\tilde{\mu}_1|\mu)$, $MSE(\tilde{\mu}_1|\mu), B(\tilde{\mu}_1|\mu), E(n|\tilde{\mu}_1), E(n|\tilde{\mu}_1)/n$, and $100(n_2/n)pr(\bar{X}_1 \in R_1)$ for testimator $\tilde{\mu}_1$ are functions of α, n_1, n_2, f, b, and λ, whereas R_1 and $pr(\bar{X}_1 \in R_1)$ are functions of α, n_1, b, and λ. We have computed these expressions for a number of values which were assigned for $f = 0.5,\ 1(1)10$, $b = 0.001, 0.01, 0.02$, $\alpha = 0.01, 0.02, 0.05, 0.01, 0.015$, and the relative variation λ takes the values $0.0(0.1)4$. This was done to provide a wide variation in the values of μ_0 around the truth. Also, from expressions (12, 17, 18, 19, 21), notice that R_2, $MSE(\tilde{\mu}_2|\mu_0)$, $B(\tilde{\mu}_2|\mu_0)$, $E(\tilde{\mu}_2|\mu_0)$, $E(\tilde{\mu}_2|\mu_0)/n$, and $pr\{\bar{X}_1 \in R_1)$ for $\tilde{\mu}_2$ are functions of α, n_1, n_2, f, and b. This was done for $\alpha = 0.01, 0.02, 0.05$, $b = 0.001, 0.01, n_1 = 5, 11$, and $n_2 = 1(1)55$. Some of these computations are given in Tables 1 to 7. We make the following observations from tables presented in this paper:

i) From the computations of relative efficiency given in Table 1, and as expected the double stage shrinkage estimators give higher relative efficiency in some region a round μ_0. It is observed that the estimator $\tilde{\mu}_1$ has smaller mean squared error than the classical single stage estimator $\bar{X}$ for the region $0 \le |\lambda| \le 3$. Thus $\tilde{\mu}_1$ may be used to improve the efficiency if the difference $\mu_0 - \mu$ is expected to belong to the effective interval (boarder range of $|\lambda|$ for which efficiency is greater than unity) $ER = [-3\sigma/\sqrt{n_1}, 3\sigma/\sqrt{n_1}]$.

ii) It is also seen that from Table 1, for fixed f, b, and α, the relative efficiency of $\tilde{\mu}_1$ is maximum when $\lambda \cong 0$ (i.e., $\mu_0 = \mu$), and much greater than the classical estimator (as much as 3500 times), whereas the relative efficiency decreases with increasing value of $|\lambda|$, and it's less than 1 for $|\lambda| > 3$ (i.e., if $(\mu_0 - \mu) \notin [-3\sigma/\sqrt{n_1}, 3\sigma/\sqrt{n_1}])$.

iii) From Tables 1 and 2, it is observed that the testimator $\tilde{\mu}_1$ is biased. The bias ratio is reasonably small if the prior point estimate μ_0 does not deviate too much from the true value μ.

iv) It is observed from our computations given in Tables 1 and 2 that the relative efficiency of $\tilde{\mu}_1$ decreases with size α of the pretest region, i.e., $\alpha = 0.01$ gives higher relative efficiency than for other values of α. As α increases, $Eff(\tilde{\mu}_1|\mu)$ remains greater than the unity, whereas for any fixed α and b, the relative efficiency is a decreasing function of n_1 when $|\lambda| \cong 0$.

v) From Table 3, the probability of avoiding the second sample is independent of n_2 and it is clearly $1 - \alpha$ at $|\lambda| = 0$ but it decreases as λ increases or n_1 increases.

Table 1: *Showing $Eff(\tilde{\mu}_1|\mu)(Ef)$ and $(\sqrt{n_1}/\mu)B(\tilde{\mu}_1|\mu)/\mu(B)$ when $f = 0.5$, and different values of b, α, and λ.*

b	α	$\|\lambda\|$	0.0	0.2	0.4	0.6	0.8	1.0	2.0	3.0
0.001	0.01	Ef	15.4028	11.975	9.933	7.132	5.632	4.733	3.325	2.304
		B	0.000	−0.176	−0.208	−0.235	−1.277	−0.295	−0.397	−0.56
	0.05	Ef	11.284	9.842	7.243	5.573	4.276	3.518	2.846	2.105
		B	0.0000	−0.154	−0.189	−0.219	−0.259	−0.285	−0.364	−0.461
	0.1	Ef	9.285	7.177	6.428	5.856	4.0165	3.627	2.354	1.913
		B	0.000	−0.138	−0.171	−0.219	−0.225	−0.251	−0.339	−0.423
	0.135	Ef	6.564	5.119	4.417	3.922	3.217	2.843	2.114	1.500
		B	0.000	−0.109	−0.145	−0.173	−0.216	−0.236	−0.314	−0.399
0.01	0.01	Ef	14.829	11.284	8.345	6.823	5.426	4.064	3.156	2.163
		B	0.0000	−0.143	−0.145	−0.229	−0.264	−0.284	−0.373	−0.489
	0.05	Ef	10.372	8.043	6.824	4.414	3.890	3.099	2.184	1.778
		B	0.0000	−0.138	−0.169	−0.201	−0.233	−0.265	−0.328	−0.425
	0.1	Ef	7.393	5.784	4.627	3.864	3.432	2.835	2.159	1.471
		B	0.000	−0.121	−0.153	−0.185	−0.216	−0.243	−0.305	−0.399
	0.135	Ef	6.383	4.926	4.361	3.896	3.171	2.785	2.023	1.314
		B	0.000	−0.098	−0.137	−0.156	−0.199	−0.216	−0.297	−0.379

Table 2: *Showing $Eff(\tilde{\mu}_1|\mu)(Ef)$ and $(\sqrt{n_1}/\mu)B(\tilde{\mu}_1|\mu)/\mu(B)$ when $\alpha = 0.01$, $b = 0.001$, and different values of f and λ.*

f	$\|\lambda\|$	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0
2	Ef	189.271	48.883	17.067	7.9723	5.4377	4.1283	2.7843	2.0900	1.0990
	B	0.000	−0.189	−0.360	−0.433	−0.471	−0.501	−0.479	−0.420	−0.399
4	Ef	280.215	57.006	19.725	8.2370	5.8850	4.3418	2.9657	1.8911	0.9940
	B	0.000	−0.198	−0.364	−0.441	−0.489	−0.501	−0.476	−0.399	−0.390
6	Ef	455.521	65.288	21.462	9.1907	5.9031	4.4310	3.0003	1.7873	0.9330
	B	0.000	−0.199	−0.364	−0.445	−0.489	−0.499	−0.447	−0.386	−0.378
8	Ef	1354.142	72.315	22.985	9.7143	6.2167	4.8733	3.1401	1.5010	0.9042
	B	0.000	−0.200	−0.365	−0.446	−0.492	−0.499	−0.428	−0.373	−0.362
10	Ef	3531.239	80.858	24.133	11.656	6.9177	5.4520	3.4213	1.0200	0.8940
	B	0.000	−0.202	−0.365	−0.446	−0.499	−0.489	−0.395	−0.358	−0.345

Table 3: *Showing $pr\{\bar{X}_1 \in R_1\}$ when $f = 0.5$, $b = 0.001$ and $\alpha = 0.01$.*

$\|\lambda\|$ / n_1	0.0	0.2	0.4	0.6	0.8	1.0	2.0	3.0
4	0.990	0.990	0.990	0.990	0.990	0.990	0.988	0.968
8	0.990	0.990	0.990	0.990	0.988	0.983	0.822	0.714
12	0.990	0.986	0.982	0.981	0.978	0.973	0.878	0.581
16	0.990	0.984	0.981	0.979	0.975	0.971	0.816	0.500
20	0.990	0.983	0.981	0.975	0.952	0.926	0.681	0.345

Table 4: *Showing* $E(n|\tilde{\mu}_1)$ *when* $\alpha = 0.01$, $b = 0.001$ *and* $n_1 = 12$.

$\lvert\lambda\rvert$ / f	0.0	0.2	0.4	0.6	0.8	1.0	2.0	3.0
0.5	12.048	12.081	12.101	12.112	12.123	12.157	12.725	14.507
1	12.096	12.162	12.212	12.229	12.249	12.318	13.455	17.020
2	13.192	12.325	12.430	12.450	12.508	12.640	14.918	22.050
3	12.289	12.488	12.643	12.673	12.770	12.691	16.380	27.074
4	12.385	12.651	12.861	12.902	13.026	13.280	17.841	32.105
5	12.481	12.814	13.077	13.131	13.284	13.602	19.302	37.131
10	12.962	13.629	14.156	14.260	14.571	15.211	26.610	62.266

Table 5: *Showing* $E(n|\tilde{\mu}_1)/n$ *when* $\alpha = 0.01$, $b = 0.001$ *and* $n_1 = 12$.

$\lvert\lambda\rvert$ / f	0.0	0.2	0.4	0.6	0.8	1.0	2.0	3.0
0.5	0.699	0.671	0.673	0.673	0.674	0.675	0.707	0.806
1	0.502	0.506	0.509	0.509	0.511	0.513	0.561	0.709
2	0.335	0.342	0.345	0.346	0.347	0.351	0.414	0.613
3	0.251	0.260	0.263	0.264	0.266	0.270	0.341	0.564
4	0.206	0.211	0.214	0.215	0.217	0.221	0.297	0.535
5	0.167	0.178	0.182	0.182	0.184	0.189	0.268	0.516
10	0.098	0.103	0.107	0.108	0.110	0.115	0.202	0.472

Table 6: *Showing* $(100x(n_2|n)$ $(pr\{\bar{X}_1 \in R_i\})$ *when* $\alpha = 0.01$, $b = 0.001$ *and* $n_1 = 12$.

$\lvert\lambda\rvert$ / f	0.0	0.2	0.4	0.6	0.8	1.0	2.0	3.0
0.5	33.066	32.872	32.730	32.7.1	32.615	32.431	29.268	19.360
1	49.599	49.315	49.089	49.052	48.824	48.649	43.905	29.048
2	66.132	65.752	65.461	65.364	65.097	64.719	58.541	38.731
3	74.398	73.975	73.641	73.583	73.372	72.973	65.861	43.569
4	79.358	78.910	79.552	78.481	78.279	77.851	70.253	46.477
5	82.665	82.193	81.827	81.752	81.540	81.089	73.179	48.412
10	90.180	89.661	89.260	89.177	88.941	88.460	79.823	52.813

vi) It is seen from Tables 4 and 5, that the expected sample size is close to n_1 when $\lambda = 0$ and increases very slowly with increases of $|\lambda|$ and f, whereas for any fixed α, b and n_1, the ratio $E(n|\tilde{\mu}_1)/n$ (which reflects the profligacy ratio in experimental units) is minimum when $|\lambda| = 0$, and decreases with increasing value of f.

vii) From Table 6, it is observed that the percentage of saving in sample is maximum when μ is close to μ_0 but it decreases as $|\lambda|$ increases. However, decreases in

Table 7: *Showing* $Eff(\tilde{\mu}_2|\mu_0)$, *when* $\alpha = 0.01, 0.5, 0.1$, $b = 0.001$, $n_1 = 5, 11$ *and n.*

	$n_1 = 5$			$n_1 = 11$		
n_2	0.01	0.05	0.1	0.01	0.05	0.1
5	231.551	54.289	30.654	197.237	49.272	29.384
8	392.692	89.978	49.476	278.389	82.727	42.833
11	598.497	134.088	71.912	373.825	109.894	62.077
14	851.458	186.612	97.728	483.797	140.599	78.452
17	1155.00	247.725	126.812	608.622	174.924	96.406
20	1513.40	317.792	159.145	748.681	212.811	115.874
23	1932.51	379.387	194.803	904.420	254.254	136.803
26	2419.42	487.322	233.950	1076.40	299.268	159.153
29	2983.30	588.691	276.841	1265.10	347.884	182.890
32	3636.01	702.929	323.835	1471.40	400.147	207.992
35	4392.22	831.902	375.403	1695.90	456.126	234.444
38	5271.40	978.029	432.157	1939.60	515.903	262.241
41	6298.92	1144.51	494.873	2203.40	579.583	291.383
44	7508.31	13335.32	564.540	2488.60	647.291	321.881

Table 8: *Showing* $E(n|\tilde{\mu}_2)(E_2)$, *and* $(100x(n_2|n)\ (pr\{\bar{X}_1 \in R_2\})\,(E_3)$
when $\alpha = 0.01$, $b = 0.001$, $n_1 = 5, 11$ *and n.*

	$n_1 = 5$		$n_1 = 11$	
n_2	E2	E3	E2	E3
5	5.050	49.500	11.050	68.063
8	8.080	38.077	11.080	57.316
11	5.110	30.938	11.110	49.500
14	5.140	26.053	11.140	43.560
17	5.170	22.500	11.170	38.893
20	5.200	19.800	11.200	35.129
23	5.230	17.679	11.230	32.029
26	5.260	15.968	11.260	29.432
29	5.290	14.559	11.290	27.225
32	5.320	13.378	11.320	25.326
35	5.350	12.376	11.350	23.674
38	5.380	11.512	11.380	22.224
41	5.410	10.761	11.410	20.942
44	5.440	10.102	11.440	19.800

percentage overall sample saved with increase in $|\lambda|$ is very slow irrespective f, e.g. for $\alpha = 0.01$, percentage sample saved is almost constant up to $|\lambda|$ as high as 0.8 even for f as high as 10.

viii) As the main purpose of a two-stage shrinkage testimator is to cut down the sample size without reducing efficiency, we shall like to study empirically the relation between efficiency, λ and $f = (n_2/n_1)$. Indeed the value of n_1 is dictated by the availability of the experimental data and the second sample n_2 can be produced whenever necessary by performing a new experiment. It is observed from our computation given in Table 2, that (for $0 \le |\lambda| \le 0.5$) the increment of the maximum increase in relative efficiency decreases with f and is between 19 % to 5.5 % approximately. The corresponding increment of increase in f (or in n) is fixed and is 100 %. Thus the choice $f \cong 4(n_2 \cong 4n_1)$, is recommended (which corresponds to maximum increment in relative efficiency).

ix) The behavioural pattern of testimator $\tilde{\mu}_2$ is similar to that of $\tilde{\mu}_1$ as for expected sample size, relative efficiency, probability of avoiding the second stage sample and the percentage of overall samples saved are concerned.

x) Testimator $\tilde{\mu}_1$ is better than that of Katti (Katti, 1962), Arnold and Al-Bayyati (Arnold and Al-Bayyati, 1970), Waiker, Schuurmann and Raghunathan (Waiker, Schuurmann and Raghunathan, 1984), Kambo, Handa and Al-Hemyari (Kambo *et al.*, 1991), and Waiker, Ratnaparkhi, and Schuurmann (Waiker *et al.*, 2001) and Ratnaparkhi *et al.*, 2001) both in terms of higher relative efficiency and boarder range of the effective interval. Also comparing these results with the Tables 1 and 5 of Al-Hemyari (Al-Hemyari, 2009) it is observed that the testimator $\tilde{\mu}_1$ performs better in the sense of higher relative efficiency for $0 \le |\lambda| \le 2$. Comparing Table 7 with the results of Al-Hemyari, 2009; Arnold and Al-Bayyati, 1970; Kambo *et al.*, 1991; Waiker *et al.*, 2001; Ratnaparkhi *et al.*, 2001 and Waiker *et al.*, 1984, it is seen that $\tilde{\mu}_2$ is also much better in terms of higher relative efficiency than the existing testimators with unknown σ^2.

6 Summary

It has been seen that the suggested general two-stage shrunken testimators have considerable gain in relative efficiency for many choices of constants involved in it. It is recommended that one should not consider the substantial gain in efficiency in isolation, but also the wider range of $|\lambda|$. It is really interesting that the proposed testimator gives high relative efficiency for small first sample (or large f), which reduces the cost of the experimentation, and also for large first sample (or small f) and for a broad range of $|\lambda|$. Accordingly, even if the experimenter has less confidence in the guessed value μ_0 (if $\bar{X}_1 \notin R$), the relative efficiency is also greater than the classical and all the existing testimators. Moreover, the efficiency of the suggested testimators can be increased considerably by choosing the scalars α, n_1, n_2 and b appropriately. Thus it is recommended to use the proposed testimators in practice.

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